CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

Discrete Mathematics 86 (1990) 27–31
North-Holland

RECENT PROBLEMS AND RESULTS ABOUT KERNELS IN DIRECTED GRAPHS*

C. BERGE** and P. DUCHET
Université de Paris, Centre National de Recherche Scientifique, Paris, France

Received 2 December 1988

In Section 1, we survey the existence theorems for a kernel; in Section 2, we discuss a new conjecture which could constitute a bridge between the kernel problems and the perfect graph conjecture. In fact, we believe that a graph is 'quasi-perfect' if and only if it is perfect.

1. Kernel-perfect graphs

Let G be a directed graph. Its vertex-set will be denoted by X, and its arcs (or 'directed edges') are a subset of the cartesian product $X \times X$. A *kernel* of G is a subset S of X which is 'stable' (*independent*, i.e.: a vertex in S has no successor in S) and 'absorbant' (*dominating*, i.e. a vertex not in S has a successor in S).

This concept has found many applications, for instance in cooperative n-person games, in Nim-type games (cf. [1]), in logic (cf. [2]), etc. So, the main question is: Which structural properties of a graph imply the existence of a kernel? By 'subgraph', we shall always mean 'induced subgraph'. A graph G whose all subgraphs have kernels is called *kernel-perfect*. Otherwise, G is *kernel-imperfect*. The classical results (see [1]) are:

(1) *A symmetric graph is kernel-perfect* (trivial);

(2) *A transitive graph is kernel-perfect, and all kernels have the same cardinality* (König);

(3) *A graph without circuits is kernel-perfect, and its kernel is unique* (von Neumann);

(4) *A graph without odd circuits is kernel-perfect* (Richardson).

Many extensions of Richardson's Theorem have been found in the last ten years. An easy one is:

Proposition 1.1. *Let G be a graph such that every odd circuit has all its arcs belonging to pairs of parallel arcs ('double-edges'). Then G is kernel-perfect.*

*Reprinted with permission from Ringeisen, Richard D., and Roberts, Fred S. (1988), Applications of Discrete Mathematics, Proceedings of the Third Conference on Discrete Mathematics, Clemson University, Clemson, South Carolina, May 14–16, 1986, SIAM Publications, Philadelphia, PA. Copyright 1988 Society for Industrial and Applied Mathematics.

** The first author gratefully acknowledges the partial support of the Air Force Office of Scientific Research under grant number AFOSR-0271 to Rutgers University.

0012-365X/90/$03.50 © 1990 — Elsevier Science Publishers B.V. (North-Holland)

Proof. It suffices to show that G possesses a kernel. Let $x_1, x_2, \ldots, x_n$ be the vertices of G. Let G' be the graph obtained from G by removing the arc (x_i, x_j) if $i > j$ and (x_i, x_j) belongs to a pair of parallel arcs. Clearly, G' has a kernel S'. In G, the set S' is both stable and absorbant. Hence S' is a kernel of G. □

Other theorems have been found recently, in particular the following:

(1) *If every odd circuit* $[x_1, x_2, \ldots, x_{2k+1}, x_1]$ *has two chords of the type* (x_i, x_{i+2}), (x_{i+1}, x_{i+3}) *then the graph is kernel-perfect* (Duchet and Meyniel [11]).

(2) *If every odd circuit has at least two arcs belonging to pairs of parallel arcs, then the graph is kernel-perfect* (Duchet [8]).

(3) *If every odd circuit has two chords whose heads are consecutive vertices of the circuit, then the graph is kernel-perfect* (Neuman–Lara and Galeana–Sanchez [13]).

However, it is false that a graph G such that all odd circuits have two chords is kernel-perfect (Neuman–Lara and Galeana–Sanchez [12]). Other related results are due to Meyniel (unpublished), or to Neuman–Lara and Galeana–Sanchez (unpublished).

A *critical kernel-imperfect* graph is a graph G without kernel such that every strict subgraph is kernel-perfect. We have:

Proposition 1.2. *A critical kernel-imperfect graph is strongly connected.*

Proof. Otherwise, let G be a critical kernel-imperfect graph which is not strongly connected. There exists a strong component C_1 of G such that no arcs go from C_1 to $X - C_1$ ('terminal component'). Let S_1 be a kernel of G_{C_1}. Consider the subgraph of G induced by $C_2 = X - S_1 - \{x \mid x \in X,\ x \text{ has a successor in } S_1\}$. Clearly, C_2 is a strict subset of X; consequently, G_{C_2} has a kernel S_2. The set $S_1 \cup S_2$ is stable, because no arc goes from S_1 to S_2 (because S_2 does not meet the terminal component C_1), and no arc goes from S_2 to S_1 (by the definition of C_2). Therefore, the set $S_1 \cup S_2$, which is also absorbant for G, is a kernel of G: a contradiction. □

Remark. This proposition yields a very simple proof for the theorem of Richardson. Let G be a graph with no odd circuits which would not be kernel-perfect. Let G' be a critical kernel imperfect subgraph of G. Since G' has no odd circuits, its vertices can be colored with two colors by the following procedure: color with blue a given vector x_0. Color with red every successor of blue vertex. Color with blue every successor of a red vertex.

Clearly, no vertex can be colored with both colors (otherwise there would be an odd circuit). By Proposition 1.2, G' is strongly connected, and therefore all its vertices will be colored when the procedure terminates. Then, the set consisting of all blue vertices is both stable and absorbant for G': a contradiction.

Various examples of critical kernel-perfect graphs exist in the literature, but no structural characterization has been found so far. However, we must keep in mind the following remark:

Proposition 1.3. *Let G be a kernel-perfect graph. Then every complete subgraph ('clique') has a vertex which is successor of all its other vertices.*

2. Quasi-perfect graphs

Let us recall that a simple graph G is *perfect* if every induced subgraph G_A satisfies $\alpha(G_A) = \Theta(G_A)$, where $\alpha(G)$ denotes the stability number of G (maximum number of independent vertices), and Θ denotes the minimum number of cliques needed to cover the vertex-set of G. The *perfect graph conjecture* is: G is perfect if and only if there is no induced odd cycle C_{2k+1} (with $k \geqslant 2$), and no induced $\bar{C}_{2k+1}$ (complement of a C_{2k+1}, $k \geqslant 2$).

Recall that an 'orientation' of an edge consists in replacing this edge either by an arc or by two parallel arcs in opposite directions; in a directed graph, the orientation is *normal* if every clique contains a vertex which is successor of all its other vertices. A simple graph G is *quasi-perfect* (or 'solvable') if every normal orientation of its edges results in a kernel-perfect directed graph.

Thus, a clique K_n is a quasi-perfect graph; furthermore, we have:

Proposition 2.1 *A complete directed graph has a normal orientation if and only if every circuit has at least one arc belonging to a pair of parallel arcs.*

Proof. Let G be a complete directed graph with a normal orientation; then a circuit μ has a vertex x which is successor of all its other vertices; therefore the arc of the circuit which is incident from x belongs to a pair of parallel arcs.

Conversely, assume that $G = (X, U)$ is a complete (directed) graph whose circuits satisfy the condition of Proposition 2.1. We shall assume that its orientation is not normal, to obtain a contradiction. Let C be a clique of G having no kernel, and let $x_1 \in C$. Since $\{x_1\}$ is not a kernel of C, there exists a vertex $x_1 \in C$ with $(x_2, x_1) \notin U$, and $(x_1, x_2) \in U$. Also, there exists a vertex $x_3 \in C$ with $(x_3, x_2) \notin U$ and $(x_2, x_3) \in U$, etc. . . . ; so we define a sequence $x_1, x_2, x_3, \ldots, x_i, \ldots$ of distinct vertices with $(x_i, x_{i+1}) \in U$, and $(x_{i+1}, x_i) \notin U$. Since the graph is finite, the sequence $(x_1, x_2, \ldots, x_q)$ terminates with x_q, and for some $p < q$, we have $(x_q, x_p) \in U$, $(x_p, x_q) \notin U$. Then the sequence $(x_p, \ldots, x_q, x_p)$ constitutes a circuit with no arc belonging to a pair of parallel arcs. The contradiction follows. □

Proposition 2.2. *The graph C_{2k+1}, with $k \geqslant 2$, is not quasi-perfect.*

Proof. The orientation $\vec{C}_{2k+1}$ of C_{2k+1} as a directed circuit is a normal orientation. Since $\vec{C}_{2k+1}$ has no kernel, the graph C_{2k+1} is not quasi-perfect. □

Proposition 2.3. *The graph $\bar{C}_{2k+1}$, with $k \geqslant 2$, is not quasi-perfect.*

Proof. Let $[x_1, x_2, \ldots, x_{2k+1} = x_1]$ be the cycle C_{2k+1}. We can provide $\bar{C}_{2k+1}$ with the following normal orientation: join x_i and x_j with two parallels arcs if $j \neq i-2,\ i-1,\ i, i+1,\ i+2$; join x_i and x_j with only one arc (x_i, x_j) if $j = i+2$. Clearly, this is a normal orientation of $\bar{C}_{2k+1}$. Furthermore, the set $\{x_i\}$ is not a kernel, because $(x_{i+2}, x_i) \notin U$; neither is the set $\{x_i, x_{i+1}\}$ because $(x_{i+2}, x_i) \notin U$. Hence, the directed graph has no kernel; so the graph $\bar{C}_{2k+1}$ is not quasi-perfect. □

Proposition 2.4. *Every induced subgraph of a quasi-perfect graph G is quasi-perfect.*

Proof. Let G_A be the subgraph of G induced by $A \subset X$; let $\vec{G}_A$ be a normal orientation of G_A. We have to show that $\vec{G}_A$ has a kernel. Let $\vec{G}$ be a normal orientation of G obtained by directing every edge $[x, y]$ as follows:

If $x, y \in A$, direct $[x, y]$ as in $\vec{G}_A$. If $x \notin A$, $y \in A$, direct $[x, y]$ from x to y. If $x \notin A$, $y \notin A$, direct $[x, y]$ in both directions.

Clearly, this is a normal orientation. Since G is quasi-perfect, $\vec{G}$ has a kernel S. Clearly, the set $S \cap A$ is a kernel of $\vec{G}_A$; this achieves the proof. □

It follows from the Propositions 2.2, 2.3, 2.4 that a quasi-perfect graph has no induced C_{2k+1} and no induced $\bar{C}_{2k+1}$ and we do not know any other minimal prohibited configuration. This justifies the name of 'quasi-perfect graph'. In the last two years, several people tried to prove the quasi-perfectness for the main classes of perfect graphs. To summarize these results, consider the following list of properties:

(1) G is chordal (triangulated): every cycle has a chord;

(2) G is weakly chordal (weakly triangulated) (Hayward): no induced C_k, $k \geqslant 5$ and no induced $\bar{C}_k$, $k \geqslant 5$;

(3) G is i-triangulated (Gallai): every odd cycle has two non-crossing chords;

(4) G is a parity graph (Olaru–Sachs): every odd cycle has two crossing chords;

(5) G is a Meyniel graph (Meyniel): every odd cycle has two chords;

(6) G is quasi-perfect;

(7) G has no induced C_{2k+1}, $k \neq 2$, and no induced $\bar{C}_{2k+1}$, $k \neq 2$;

(8) G is perfect.

It is well known (see [1]) that $(1) \Rightarrow (2) \Rightarrow (8)$, or $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (8)$, or $(4) \Rightarrow (5) \Rightarrow (8)$.

Maffray [15] has proved that $(1) \Rightarrow (6)$; it follows from Jacob [14] and Maffray [15] that $(3) \Rightarrow (6)$. We do not know if $(5) \Rightarrow (6)$, or if $(2) \Rightarrow (6)$.

Is it true that (8) $\Rightarrow$ (6)? (Conjecture of Berge–Duchet [3]).
Is it true that (7) $\Rightarrow$ (8)? (Perfect Graph Conjecture).
Is it true that (6) $\Rightarrow$ (8)? (Weak form of the Perfect Graph Conjecture).

Is it true that the odd circuits are the only connected kernless graphs such that the removal of any arc results in a graph with a kernel? (Conjecture of Duchet, [9])

References

[1] C. Berge, Graphs, North-Holland Mathematical Library, Vol. 6 (North-Holland, Amsterdam, 1985) Chapter 14.
[2] C. Berge, Nouvelles extensions du noyau d'un graphe et ses applications en théorie des jeux, Publ. Econométriques 6 (1977).
[3] C. Berge and P. Duchet, Problème, Séminaire MSH, 1983.
[4] C. Berge and A. Ramachandra Rao, A combinatorial problem in logic, Discrete Math. 17 (1977) 23–26.
[5] C. Berge and P. Duchet, Perfect graphs and kernels, Bull. Inst. Math. Acad. Sinica 16 (1988) 263–274.
[6] M. Blidia, Contribution à l'étude des noyaux dans les graphes, Thesis, Paris, 1984.
[7] M. Blidia, Every parity digraph has a kernel, Combinatorica 6 (1986) 23–27.
[8] M. Blidia, P. Duchet and F. Maffray, On kernel of perfect graphs, Rutcor Res. Report 4–88, 1988.
[9] P. Duchet, Graphes noyaux parfaits, Ann. Discrete Math. 9 (1980) 93–101.
[10] P. Duchet and H. Meyniel, A note on kernel-critical graphs, Discrete Math. 22 (1981) 103–105.
[11] P. Duchet and H. Meyniel, Une généralisation du théorème de Richardson sur l'existence de noyaux dans le graphes orientés, Discrete Math. 43 (1983) 21–27.
[12] H. Galeana-Sanchez and V. Neuman-Lara, A counterexample to a conjecture of Meyniel on kernel-perfect graphs, Discrete Math. 41 (1982) 105–107.
[13] H. Galeana-Sanchez and V. Neuman-Lara, On kernels and semikernels of diagraphs, Discrete Math. 48 (1984) 67–76.
[14] H. Jacob, Etude théorique du noyau d'un graphe, Thesis, Paris 6, 1979.
[15] F. Maffray, Sur l'existence des noyaux dans les graphes parfaits, Thesis, Paris 6, 1984.
[16] F. Maffray, On kernels in i-triangulated graphs, Discrete Math. 61 (1986) 247–251.